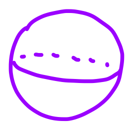
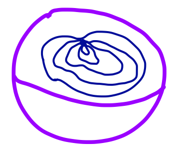
I. Fundamental Group and Covering Spaces

A. Fundamental Group

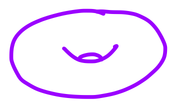
the basic idea is to "probe the topology of a space with loops mapped into the space"

intuitively any loop in  S^2 can be "shrunk to a point"

↖ i.e. homotoped to a constant loop

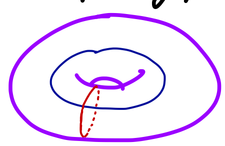


but there are loops in

$T^2 =$ 

that get "caught on the topology" and cannot be shrunk.

↗ find "holes" in the space



rigorously, as we said above the fundamental group of a topological space X with a base point $x_0 \in X$ is

$$\pi_1(X, x_0) = [S^1, X]_0$$

homotopy classes of based maps from S^1 to X

we want to see a group structure on this, to this end we need

exercise: let $S^1 \subset \mathbb{R}^2$ be the unit circle

$$p : [0,1] \to S^1$$
$$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

is a quotient map (i.e. can think of S^1 as $[0,1]$ with end pts identified)

moreover, there is a one-to-one correspondence

between $\quad \gamma : ([0,1], \{0,1\}) \to (X, x_0)$ call this a based loop in X

and

$$\hat{\gamma} : (S^1, \{(1,0)\}) \to (X, x_0)$$

(given by $\hat{\gamma} \circ p = \gamma$)

so $[S^1, X]_0$ is the same as $[([0,1],\{0,1\}), (X, x_0)]$ homotopy, rel end pts, classes of loops in X based at x_0

if $\gamma : [0,1] \to X$ a based loop, then its homotopy class is denoted $[\gamma]$

if γ_1, γ_2 are two loops based at x_0 then define

$\gamma_1 * \gamma_2$ to be the loop

$$\gamma_1 * \gamma_2 : [0,1] \to X : t \longmapsto \begin{cases} \gamma(2t) & 0 \le t \le 1/2 \\ \gamma(2t-1) & 1/2 \le t \le 1 \end{cases}$$

i.e. go around γ_1 then around γ_2

$\gamma_1 * \gamma_2$ is clearly well-defined on loops, but is it well-defined on homotopy classes of loops?

let $\gamma_1 \sim \gamma_2$ by homotopy $H : [0,1] \times [0,1] \to X$

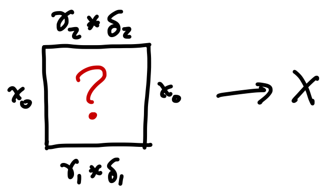
$\delta_1 \sim \delta_2$ " " $G : [0,1] \times [0,1] \to X$

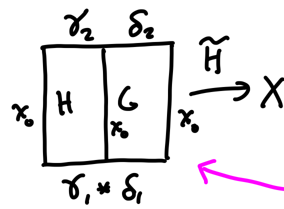
we need to find a homotopy $\gamma_1 * \delta_1$ to $\gamma_2 * \delta_2$

that is a map $\tilde{H} : [0,1] \times [0,1] \to X$ s.t. $\tilde{H}(t,0) = \gamma_1 * \delta_1$

$\tilde{H}(t,1) = \gamma_2 * \delta_2$

$\tilde{H}(i,s) = x_0 \quad i = 0,1, \forall s$

i.e. 

we can fill in ? with H and G: 

can use such pictures to get idea for homotopy

rigorously $$\tilde{H}(t,s) = \begin{cases} H(2t, s) & 0 \le t \le 1/2 \\ G(2t-1, s) & 1/2 \le t \le 1 \end{cases}$$

so $[\gamma_1] * [\delta_1] = [\gamma_1 * \delta_1]$ is well-defined!

<u>lemma 1:</u>

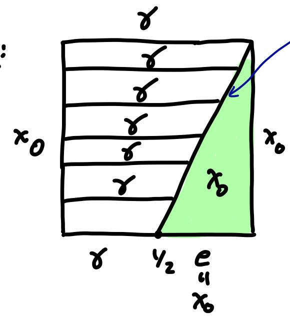
$\boxed{(\pi_1(X, x_0), *) \text{ is a group}}$

Proof:

identity: let $e:[0,1]\to X : t \mapsto x_0$ constant loop

note: $[e]*[\gamma] = [\gamma] = [\gamma]*[e]$

Picture:

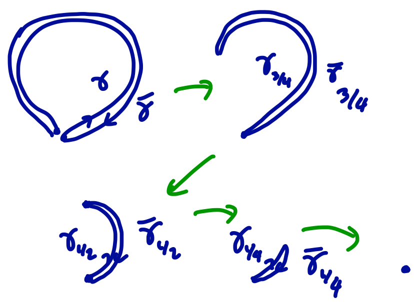


this line is $s = 2t-1$, $t = \frac{s+1}{2}$

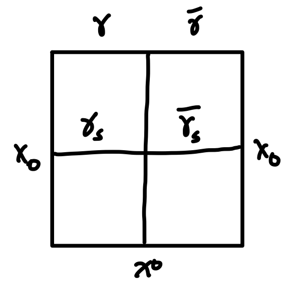
translate:

$$H(t,s) = \begin{cases} \gamma\left(\frac{2t}{1+s}\right) & 0 \le t \le \frac{1+s}{2} \\ x_0 & \frac{1+s}{2} \le t \le 0 \end{cases}$$

inverses: given $[\gamma]$, then $[\gamma]^{-1} = [\bar\gamma]$ where $\bar\gamma(t) = \gamma(1-t)$

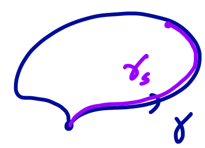


Picture:



where $\gamma_s(t) = \gamma(st)$

note $\bar\gamma_s(t) = \gamma(s-st)$

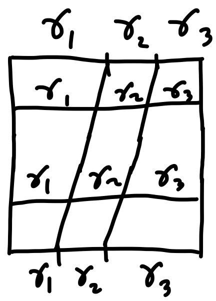


rigorously

$$H(t,s) = \begin{cases} \gamma_s(2t) & t \le 1/2 \\ \bar\gamma_s(2t-1) & t \ge 1/2 \end{cases} = \begin{cases} \gamma(2st) & t \le 1/2 \\ \gamma(s - s(2t-1)) & t \ge 1/2 \end{cases}$$

associativity: need to see $(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$

Picture:



exercise: write out H.

If $f: X \to Y$ a map
$x_0 \in X$ any $y_0 = f(x_0)$

then given any loop $\gamma:[0,1]\to X$ based at x_0

we get a loop $f\circ\gamma : [0,1] \to Y$ based at y_0

exercise: If $\gamma \sim \delta$ then $f\circ\gamma \sim f\circ\delta$

so f induces a map

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [f \circ \gamma]$$

lemma 2:

$\boxed{f_* \text{ is a homomorphism}}$

Proof: $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$

$$\gamma_1 * \gamma_2 (t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

$$(f\circ\gamma_1) * (f \circ \gamma_2) = \begin{cases} f\circ\gamma_1(2t) & 0 \le t \le 1/2 \\ f\circ\gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

so $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$

i.e. $f_*([\gamma_1] * [\gamma_2]) = f_*([\gamma_1]) * f_*([\gamma_2])$ ▦

exercise:

1) $(f \circ g)_* = f_* \circ g_*$

2) if $f: X \to Y$ is homotopic to $g: X \to Y$ relative to $x_0 \in X$ then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

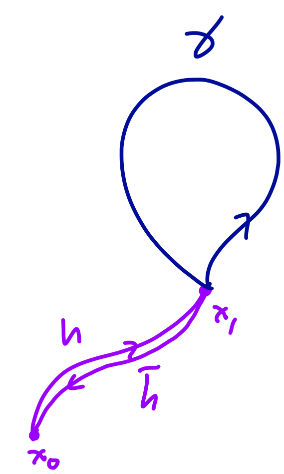
How does π_1 depend on the base point?

let $h: [0,1] \to X$ be a path with $h(0) = x_0$ and $h(1) = x_1$

if γ is a loop in X based at x_1, then note

$$h * \gamma * \bar{h} (t) = \begin{cases} h(3t) & 0 \le t \le 1/3 \\ \gamma(3t-1) & 1/3 \le t \le 2/3 \\ \bar{h}(3t-1) & 2/3 \le t \le 1 \end{cases}$$

is a loop based at x_0



lemma 3:

$$\boxed{\begin{array}{l} h \text{ induces an isomorphism} \\ \phi_h : \pi_1(X, x_1) \to \pi_1(X, x_0) \\ [\gamma] \longmapsto [h * \gamma * \bar{h}] \end{array}}$$

Remarks:

1) so isomorphism type of $\pi_1(X,x_0)$ only depends on path component of X in which x_0 lies
2) The isomorphism depends on h !

Proof: ϕ_h is a well-defined homomorphism (exercise)

Claim $\phi_{\bar h}$ is the inverse of ϕ_h

indeed given $[\gamma] \in \pi_1(X, x_0)$

$$\phi_h \circ \phi_{\bar h}([\gamma]) = [\underbrace{h * \bar h}_{} * \gamma * \underbrace{h * \bar h}_{}]$$

loop based at x_0

$$= [h * \bar h] * [\gamma] * [h * \bar h]$$

but $h * \bar h \sim e$ as a loop based at x_0

so $\phi_h \circ \phi_{\bar h}([\gamma]) = [e] * [\gamma] * [e] = [\gamma]$

you can similarly check $\phi_{\bar h} \circ \phi_h = id_{\pi_1(X,x_1)}$

Th^m 4:

If $f: X \to Y$ is a homotopy equivalence, then
$$f_*: \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$
is an isomorphism

to prove this we need

lemma 5:

Suppose $f_0, f_1: X \to Y$ are homotopic via the homotopy
$$H: X \times [0,1] \to Y$$
let $x_0 \in X$ and $h: [0,1] \to Y : t \longmapsto H(x_0, t)$

Then
$$\begin{tikzcd} & \pi_1(Y, f_0(x_0)) \\ \pi_1(X, x_0) \arrow[ur, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] \arrow[r, phantom, "\circ"] & {} \\ & \pi_1(Y, f_1(x_0)) \arrow[uu, "\phi_h"'] \end{tikzcd}$$

Proof of Th^m 4:

let g be the homotopy inverse of f so

$$\pi_1(X,x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

now $g_* \circ f_* \sim id_X$ so by lemma $\exists$ path h st.

$$g_* \circ f_* = \phi_h \quad \text{an isomorphism}$$

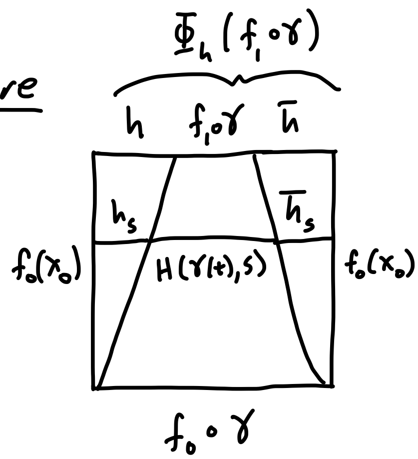
so f_* is injective

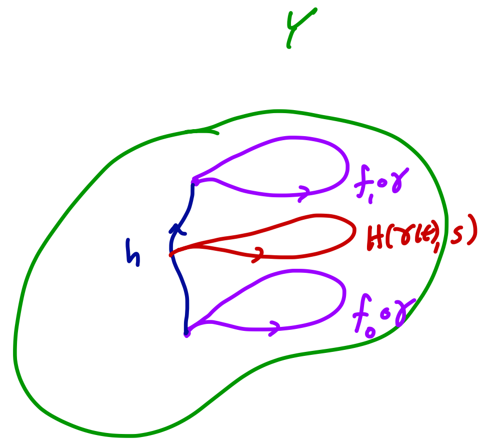
similarly $f_* \circ g_*$ is an isomorphism so f_* is surjective

$\therefore$ f_* an isomorphism ▦

Proof of lemma 5:

Picture





exercise: write out explicit homotopy ▦

Recap: We have a "functor"

$$\left\{\begin{array}{c}\text{pointed topological} \\ \text{spaces,} \\ \text{pointed maps}\end{array}\right\} \Longrightarrow \left\{\begin{array}{c}\text{groups,} \\ \text{homomorphisms}\end{array}\right\}$$

homotopic spaces map to isomorphic groups

homotopic functions map to the "same" homomorphism

B. Simple Computations

lemma 6:

> If X is contractible
> then $\pi_1(X, x_0) = \{1\}$ $\forall x_0 \in X$

Proof: If X is a one point space, then $\exists!$ loop $\gamma : [0,1] \to X$ (constant loop)
so $\pi_1(X, x_0) = \{1\}$

X contractible $\Rightarrow$ $X \simeq$ one point space
so done by $Th^{\underline{m}}$ 4

Remark: A space X is called simply connected
if 1) X is path connected, and
2) $\pi_1(X, x_0) = \{1\}$

so contractible spaces are simply connected

simply connected means "path connected in a particular simple way"

lemma 7:

> X is simply connected $\iff$ every two points in X are connected by a unique homotopy class of paths in X

Proof: ($\Leftarrow$) path connected by existence of path
any loop based at x_0 homotopic to constant loop by uniqueness of homotopy class of path ✓

($\Rightarrow$) path connected gives existence of path a to b
given 2 paths $\gamma, \delta : [0,1] \to X$ from a to b
simple connectivity implies $a * \bar{b} \sim e_a$ ← constant a path

now $\gamma \sim \gamma * (\bar{\delta} * \delta) \sim (\gamma * \bar{\delta}) * \delta \sim e_a * \delta \sim \delta$

from proof of lemma 1 even though δ a path $\bar{\delta} * \delta \sim e_b$

from proof of lemma 1

(all homotopies rell end pts of path)

Th^{m} 8:

$$\boxed{\pi_1(S^n) = \{1\} \ \forall n \geq 2}$$

need lemma

lemma 9:

let $X = A \cup B$

A, B, and $A \cap B$ open and path connected

$x_0 \in A \cap B$

Then any loop $\gamma : [0,1] \to X$ based at x_0

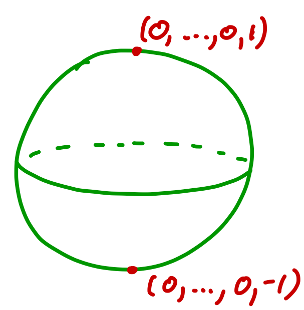
can be written as $\gamma \sim \gamma_1 * \ldots * \gamma_n$

where each γ_i is a loop in A or B based at x_0

Proof of Th^{m} 8:

let $A = S^n - \{(0,\ldots,0,1)\} \cong \mathbb{R}^n$

$B = S^n - \{(0,\ldots,0,-1)\} \cong \mathbb{R}^n$



$A \cap B = S^n - \left\{\begin{matrix}(0,\ldots,-1)\\(0,\ldots,1)\end{matrix}\right\} \cong S^{n-1} \times \mathbb{R}$

all are path connected

take $x_0 \in A \cap B$

any $[\gamma] \in \pi_1(S^n, x_0)$ can be written as

$[\gamma] = [\gamma_1][\gamma_2]\ldots[\gamma_n]$ where $[\gamma_i] \in \pi_1(A, x_0)$ or $\pi_1(B, x_0)$

by lemma 9

but $\pi_1(A,x_0)=\{1\}=\pi_1(B,x_0)$ so $[\gamma]=[e_{x_0}]$

and hence $\pi_1(S^n,x_0)=\{1\}$ $\blacksquare$

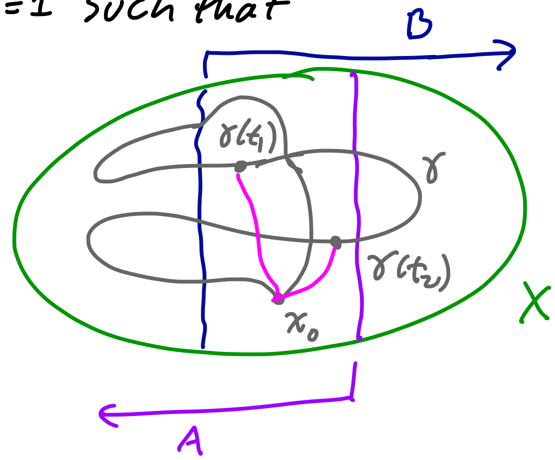
Proof of lemma 9:

given $\gamma:[0,1]\to X$ a loop based at x_0

<u>Claim</u>: there exist $0=t_0<t_1<\dots<t_n=1$ such that

$$\operatorname{im}\gamma|_{[t_{i-1},t_i]}\subset A \text{ or } B$$

and $\gamma(t_i)\in A\cap B$



given this let $\delta_i:[0,1]\to A\cap B$ connect x_0 to $\gamma(t_i)$

and $\gamma_i=\gamma|_{[t_{i-1},t_i]}$

<u>note</u>: $\gamma\sim\gamma_1*\gamma_2*\dots*\gamma_n\sim\underbrace{(\gamma_1*\bar{\delta}_1)}_{\text{loop in } A \text{ or } B}*\underbrace{(\delta_1*\gamma_2*\bar{\delta}_2)}_{\text{loop in } A \text{ or } B}*\delta_2\cdots\underbrace{(\delta_{n-1}*\gamma_n)}_{\text{loop in } A \text{ or } B}$ ✓

<u>Pf of Claim</u>:

need <u>Topology Fact (Lebesgue number lemma)</u>

Hatcher's proof implicitly uses the Axiom of choice (when choosing intervals containing each point). The proof here is more "direct"

for a proof see any topology or analysis book

X a compact metric space
$\{U_\alpha\}_{\alpha\in A}$ an open cover
$\exists$ a number $\delta>0$ (Lebesgue #)
s.t. $\forall$ sets S with $\operatorname{diam}(S)<\delta$
$\exists\alpha$ s.t. $S\subset U_\alpha$

now $U_1=\gamma^{-1}(A)$, $U_2=\gamma^{-1}(B)$ is an open cover of $[0,1]$ so $\exists\,\delta>0$ s.t. if $|b-a|<\delta$ then $[a,b]\subset U_i$ $i=0$ or 1

let n be s.t. $\frac{1}{n} < \delta$

now $\gamma|_{[\frac{i}{n}, \frac{i+1}{n}]} \subset A$ or B

so start with $t_i = \frac{i}{n}$ $\quad i = 0, \dots n$

now if $\gamma_{[t_{i-1}, t_i]}, \gamma_{[t_i, t_{i+1}]}$ both in A or B

the throw out t_i

continuing gives desired partition $\blacksquare$

Th^m 10:

$$\boxed{\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)}$$

Proof:

$$\Phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$$
$$([\gamma], [\delta]) \longmapsto [\gamma \times \delta]$$

where $(\gamma \times \delta)(t) = (\gamma(t), \delta(t))$

is an isomorphism

exercise: 1) show Φ is well-defined homomorphism

2) show Φ is bijection (use projection) $\blacksquare$

C. Fundamental Group of S^1

Th^m 11:

$\pi_1(S^1, (1,0)) \cong \mathbb{Z}$

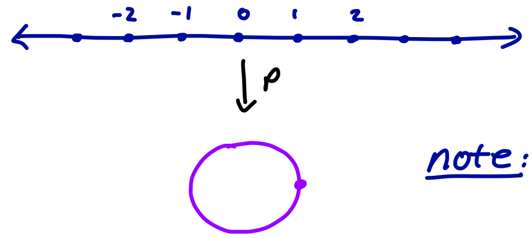
the isomorphism sends $n \in \mathbb{Z}$ to

$\gamma_n : [0,1] \to S^1 : t \mapsto (\cos 2n\pi t, \sin 2n\pi t)$

Remark: Proof is an example of very important technique that we will see again!

The proof involves studying the map

$$p : \mathbb{R} \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$$



note: $p^{-1}((1,0)) = \mathbb{Z}$

p is a special case of a covering map (we will study these quite a bit later)

If $\gamma : [0,1] \to S^1$ is a path based at $(1,0)$ then a lift of γ based at $n \in \mathbb{Z}$ is a map $\tilde{\gamma} : [0,1] \to \mathbb{R}$ s.t.

1) $\tilde{\gamma}(0) = n$

2) $p \circ \tilde{\gamma}(x) = \gamma(x) \quad \forall x$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] \arrow[ur, phantom, "\circ" near end] & S^1 \end{tikzcd}$$

lemma 12:

path lifting →

a) for each $n \in \mathbb{Z}$, each loop $\gamma : [0,1] \to S^1$ based at $(1,0)$ lifts to a unique path $\tilde{\gamma}_n$ based at n.

homotopy lifting →

b) if $\gamma \sim \gamma'$ are loops in S^1 based at $(1,0)$ and $\tilde{\gamma}_n$ and $\tilde{\gamma}'_n$ are their lifts based at n, then $\tilde{\gamma}_n \sim \tilde{\gamma}'_n$ rel $\{0,1\}$

Proof of Th^m 11 given lemma 12:

Given $\gamma \in [\gamma] \in \pi_1(S^1, (1,0))$

lemma 12 a) says $\exists! \; \tilde{\gamma}_0 : [0,1] \to \mathbb{R}$

since $\tilde{\gamma}_0(1) \in p^{-1}((1,0)) = \mathbb{Z}$ we can define

$$\Phi : \pi_1(S^1, (1,0)) \to \mathbb{Z}$$
$$[\gamma] \mapsto \tilde{\gamma}_0(1)$$

lemma 12 b) say Φ is well-defined

Φ surjeitive: let $\tilde{\delta}^n(t) = nt$ for $t \in [0,1]$

and $\delta^n(t) = p \circ \tilde{\delta}^n$

clearly $\tilde{\delta}^n$ is a lift based at 0 of the loop δ^n

and $\Phi([\delta_n]) = n$ ✓

Φ is injective:

suppose γ, γ' are two loops in S^1 s.t. $\tilde{\gamma}_0(1) = \tilde{\gamma}_0'(1)$

set $\tilde{H}(s,t) = (1-t)\tilde{\gamma}_0(s) + t\,\tilde{\gamma}_0'(s)$

and $H(s,t) = p \circ \tilde{H}(s,t)$

note: $H(s,0) = \gamma(s)$

$H(s,1) = \gamma'(s)$

$H(0,t) = (1,0) = H(1,t) \quad \forall t$

i.e. H is a homotopy of based loops

i.e. $\gamma \sim \gamma'$ ✓

Φ a homomorphism:

given $[\gamma], [\gamma'] \in \pi_1(S^1, (1,0))$

let $\tilde{\gamma}_0, \tilde{\gamma}_0'$ be the lifts of γ, γ' (based at 0)

$\Phi([\gamma]) = \tilde{\gamma}_0(1) = n \qquad \Phi([\gamma']) = \tilde{\gamma}_0'(1) = m$

note: 1) $\tilde{\gamma}_n'(t) = n + \tilde{\gamma}_0'(t)$ since rt. hand side is a lift and lift is unique

2) $\tilde{\gamma}_0 * \tilde{\gamma}_n'$ is a lift of $\gamma * \gamma'$ based at 0

so $\Phi([\gamma][\gamma']) = \widetilde{\gamma * \gamma'}(1) = \tilde{\gamma}_0 * \tilde{\gamma}_n'(1) = n+m$

$= \Phi([\gamma]) + \Phi([\gamma'])$ ∎

Proof of lemma 12:

exercise: think about uniqueness (we will do more about this for general covering space)

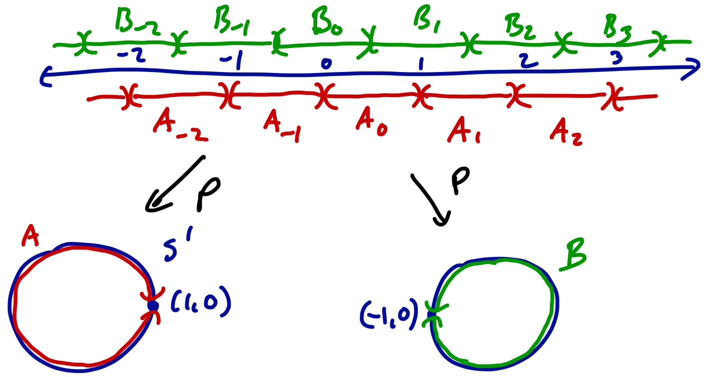
part a): let $A = S^1 - \{(1,0)\}$

$$p^{-1}(A) = \bigcup_{i \in \mathbb{Z}} \underbrace{(i, i+1)}_{A_i}$$

note: $p|_{A_i} : A_i \to A$ a homeomorphism!

similarly if $B = S^1 - \{(-1,0)\}$

then $p^{-1}(B) = \bigcup_{i\in\mathbb{Z}} \underbrace{(i-1/2,\, i+1/2)}_{B_i}$



and $p|_{B_i}: B_i \to B$ a homeomorphism

Obvious but important observation:

If $f: X \to S^1$ has image in A

then after choosing $n \in \mathbb{Z}$ $\exists$ a unique map

$$\tilde{f}: X \to A_n \subset \mathbb{R}$$

such that $p \circ \tilde{f} = f$

i.e. just set $\tilde{f} = (p|_{A_n})^{-1} \circ f$

similarly for $f(X) \subset B$.

now given a loop $\gamma: [0,1] \to S^1$ based at $(1,0)$

<u>note:</u> $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ an open cover of compact $[0,1]$

so $\exists$ Lebesgue number $\delta > 0$ for cover

choose n st. $\frac{1}{n} < \delta$

note: if $t_i = \frac{i}{n}$ then $\gamma([t_i, t_{i+1}]) \subset A$ or B

if $\gamma([t_{i-1}, t_i])$ and $\gamma([t_i, t_{i+1}])$ lie in same A or B then

discard t_i (do this inductively on i)

so we get a partition $t_0 = 0 < t_1 < \dots < t_k = 1$ of $[0,1]$

st. $\gamma([t_i, t_{i+1}]) \subset \begin{cases} B & \text{for } i \text{ even} \\ A & \text{for } i \text{ odd} \end{cases}$ (since $\mathbb{Z} \cap A = \emptyset$)

now set $\tilde{\gamma}_n = (p|_{B_n})^{-1} \circ \gamma$ on $[t_0, t_1]$

note: $\tilde{\gamma}_n(t_1) \in A_k$ some k

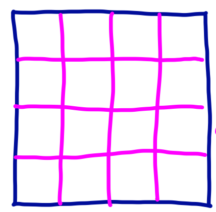
so set $\tilde{\gamma}_n = (p|_{A_k})^{-1} \circ \gamma_n$ on $[t_1, t_2]$

inductively continue to get $\tilde{\gamma}_n$ defined on $[0,1]$

since all lifts agree at endpoints $\tilde{\gamma}_n$ is continuous and clearly the desired lift!

part b):

just like proof path lifting: Given homotopy $H:[0,1]\times[0,1]\to S^1$ get Lebesgue number $\delta>0$ for $\{H^{-1}(A), H^{-1}(B)\}$ pick n s.t. $\frac{\sqrt{2}}{n}<\delta$ then consider



each square can be lifted

exercise: write out details

Many corollaries of this computation, eg

Cor 13: $\boxed{\text{There is no retraction } D^2\to\partial D^2}$

Proof: If there were a retraction $r: D^2\to S^1=\partial D^2$ then consider the inclusion map $i: S^1\to D^2$ (as ∂D^2)

note $r\circ i: S^1\to S^1$ is the identity map!

so $r_*\circ i_* = (r\circ i)_* = (id_{S^1})_* = id_{\pi_1(S^1,(1,0))}: \mathbb{Z}\to\mathbb{Z} \Rightarrow i_*$ injective

but $\pi_1(D^2,(1,0))=\{1\}$, so $i_* =$ constant map $\cancel{\text{injectivity}}$

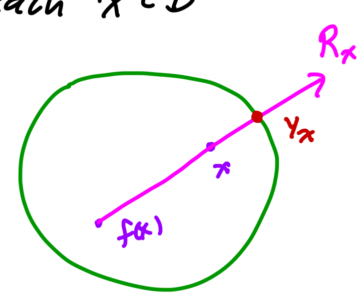
Cor 14: $\boxed{\text{any map } f: D^2\to D^2 \text{ has a fixed point}}$

Proof: if $f: D^2\to D^2$ had no fixed points, then for each $x\in D$

let $R_x =$ ray starting at $f(x)$ going through x

note: $R_x\cap\partial D^2$ is exactly 1 point y_x

set $g(x)=y_x$



exercise: $g(x)$ continuous (eq^n for R_x continuous in x
$\therefore$ eq^n for $R_x \cap S^1$ continuous in x)

cleary g a retraction! $\cancel{C}$ Cor 13

Many other applications!

1) Fundamental Th^m of Algebra
2) Borsuk-Ulam Th^m (abt maps $S^2 \to S^1$ and $S^2 \to \mathbb{R}^2$)
3) Ham sandwich th^m

⋮

see Hather's Book and supplment class webpage